The Limits of Quintessence

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1 Introduction

In this summary we will briefly talk about the dynamics of a cosmology dominated by a homogenous scalar field, with the intention of using it to model dark energy. We will begin with a brief discussion of the dynamics, followed by numerical computations for certain physically motivated testpotentials showing freezing and thawing effects, and conclude with a discussion on which combination of cosmological observables would be relevant to uncover the underlying physics of the scalar, leading us to argue that the number of interest will be $\frac{w'}{1-w}$.

2 Basics of cosmological scalar fields

We start with the Lagrangian $\mathcal{L}_Q = -\frac{1}{2}Q_{,\mu}Q^{\mu}_{,} - V(Q)$ on an FRW metric background. The stress energy tensor is then given by

$$\begin{aligned} T_{\mu\nu} &= -Q_{,\mu}\frac{\partial \mathcal{L}_Q}{\partial Q_{,\nu}} + \eta_{\mu\nu}\mathcal{L}_Q \\ &= Q_{,\mu}Q_{,\nu} + \eta_{\mu\nu}\mathcal{L}_Q \end{aligned}$$

We see that if we define $\sqrt{\rho_Q + p_Q}u_\mu = Q_{,\mu}$ and $p_Q = \mathcal{L}_Q$, we see that this is just the stress energy tensor of a perfect fluid. Imposing $u^\mu u_\mu = 1$ then gives us

$$\begin{aligned} p_Q &= -\frac{1}{2}Q_{,\mu}Q^{\mu}_{,} - V(Q) \\ \rho_Q &= Q^{\mu}_{,}Q_{,\mu} - p \\ &= -\frac{1}{2}Q_{,\mu}Q^{\mu}_{,} + V(Q) \end{aligned}$$

Imposing spatial homogeneity then gives us equation $\rho_Q = \frac{1}{2}\dot{Q}^2 + V(Q)$ and $p_Q = \frac{1}{2}\dot{Q}^2 - V(Q)$.

The equation of motion of the scalar field is given by

$$\begin{aligned} 0 &= \nabla_\mu \frac{\partial \mathcal{L}_Q}{\partial Q_{,\mu}} - \frac{\partial L}{\partial Q} \\ &= -\nabla_\mu Q^{\mu}_{,} + V_{,Q} \\ &= -Q^{\mu}_{,\mu} - \Gamma^{\mu}_{\mu\lambda}Q^{\lambda}_{,} + V_{,Q} \end{aligned}$$

again imposing homogeneity and using the fact that $\Gamma^{i}_{j0} = \delta^{i}_{j}H$ then gives us $\ddot{Q} + 3H\dot{Q} + V_{,Q} = 0$.

Starting with the density and equation of state we can go back to express the field variables as

$$
\begin{aligned}
V &= \frac{\rho - p}{2} = \frac{1}{2}(1-w)\rho \\
Q &= \int \mathrm{d}a \frac{\mathrm{d}t}{\mathrm{d}a}\frac{\mathrm{d}Q}{\mathrm{d}t} \\
&= \int \mathrm{d}a \frac{1}{aH}\sqrt{\rho + p} \\
&= \int \mathrm{d}a \frac{\sqrt{1+w}}{aH}\sqrt{\rho}
\end{aligned}
$$

3 Freezing and thawing

Given the equation of motion $\ddot{Q} + 3H\dot{Q} = -dV/dQ$, the field is inclined to minimize the potential V. The evolution of Q can be described by two models, thawing and freezing, which are determined by the morphology of $V(Q)$. If a minimal $V(Q) = 0$ is accessible with a finite field Q, the potential that is initially deviated from the stationary point with $w_0 \approx -1$ will start to thaw and roll down towards $w = 0$ as the Hubble constant decays. In this case, w slowly increases and becomes less negative or equivalently the factor $w' = \dot{w}/H = dw/d\ln a$ is positive. Since the decay of H limits the field acceleration, the equation of motion implies the upper constraints $w' < 3(1+w)$. There exist a lower bound of w' too, considering the reality that the present universe is not completely dominated by Λ and the density parameter is $\Omega_\Lambda \simeq 0.7$. Several studies, such as pseudo Nambu-Goldstone boson (PNGB) and polynomial potential, suggest a lower limit $w' > (1+w)$. One caveat is that these limits are valid for $w \leq -0.8$. Different from the thawing model, freezing model rises when the minimum of $V(Q)$ can not be achieved over a finite range of Q. The field in a freezing model gradually rolls down and decelerate so that $\dot{Q} \to 0$ and the equation-of-state parameter $w = (\frac{1}{2}\dot{Q}^2 - V)/(\frac{1}{2}\dot{Q}^2 + V) \to -1$. The steepness of the potential limits the deceleration of the field, say $\ddot{Q} > dV/dQ$, which leads to $w' > 3w(1+w)$. As for the the upper limit, an empirical bound $w' \leq 0.2w(1+w)$ is given by predictions of various models. Moreover, this limit is not definite and is applicable only for $w \leq 0.8$.

Now, we attempt to derive equations governing the evolution of w and further reproduce the figures in the $w - w'$ phase space. Using the definition of w and w', we have

$$
w' = \frac{1}{H}\frac{dw}{dt} = \frac{dw}{d\ln a} = \frac{1}{H}\frac{2V\dot{Q}\ddot{Q} - V_{,Q}\dot{Q}^3}{(\frac{1}{2}\dot{Q}^2 + V)^2}.
$$

Recalling $\ddot{Q} = -3H\dot{Q} - V_{,Q}$, $\rho_\Lambda(1+w) = \dot{Q}^2$ and $2V = \rho_\Lambda(1-w)$, we eliminate $\ddot{Q}, \dot{Q}$ in the equation and finally obtain

$$
\begin{aligned}
w' = \frac{dw}{d\ln a} &= -3(1-w^2) - \frac{dV}{dQ}\frac{1}{V}(1-w)\sqrt{\frac{\rho_\Lambda}{H^2}(1+w)} \\
&= -3(1-w^2) - \frac{dV}{dQ}\frac{M_p}{V}(1-w)\sqrt{\frac{3}{8\pi}\Omega_\Lambda(1+w)} \\
&= (w-1)[3(1+w) - \lambda\sqrt{3(1+w)\Omega_\Lambda}]
\end{aligned} \tag{1}
$$

where $\Omega_\Lambda = \rho_\Lambda/\rho_{cr}$, $\lambda = -\sqrt{\frac{1}{8\pi}\frac{dV}{dQ}\frac{M_p}{V}}$, the Planck mass $M_p = 1/\sqrt{G}$. In our calculation, we assume Ω_Λ and Λ are time-dependent (or a-dependent). The equations describing λ and Ω_Λ are

$$
\begin{aligned}
\frac{d\Omega_\Lambda}{d\ln a} &= -3(w - w_m)\Omega_\Lambda(1-\Omega_\Lambda) \\
\frac{d\lambda}{d\ln a} &= -\sqrt{3(1+w)\Omega_\Lambda}(\Gamma - 1)/\lambda^2
\end{aligned} \tag{2}
$$

where $\Gamma = VV_{,QQ}/V_{,Q}^2$ and $w_m = 0$ and $1/3$ correspond to current universe and radiation-dominated epoch respectively.

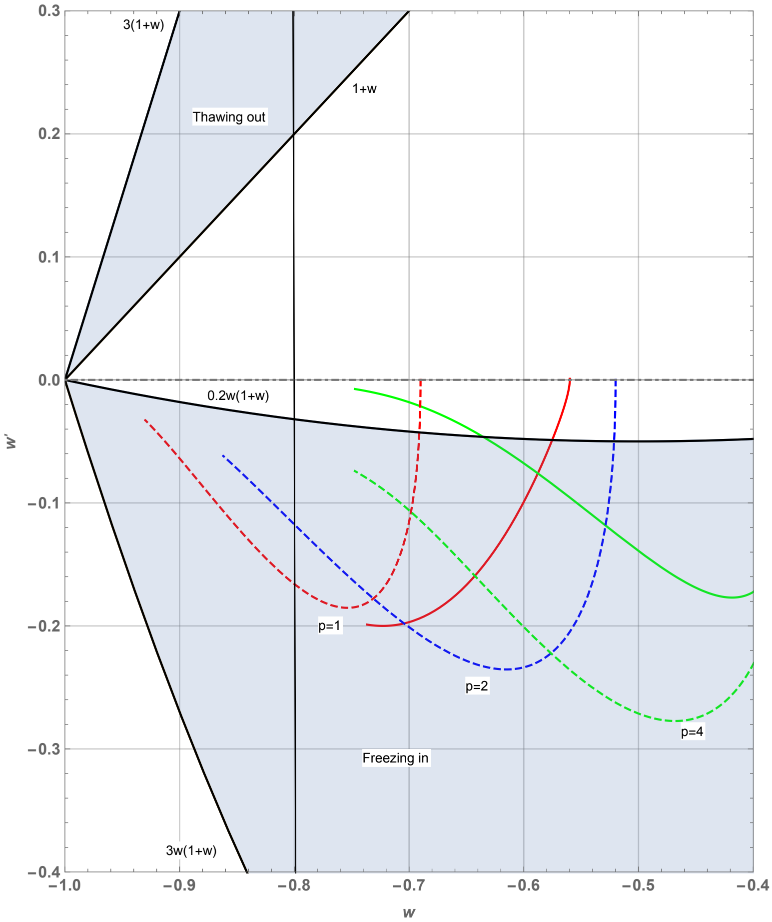

Figure 1: The evolution of w in $w - w'$ phase space. Black solid lines are the boundaries for thawing and freezing models. The vertical black line shows the maximal w that the boundary functions, e.g. $3w(1+w)$, are valid. Models in the present universe ($\Omega_\Lambda = 0.7, w_m = 0$) and radiation-dominated universe ($\Omega_\Lambda = 0.01, w_m = 1/3$) are illustrated by dashed and dot-dashed lines. Different colors correspond to the index $p = 1, 2, 4$ in the potential $V = M^{4+p}Q^{-p}$.

To find the fixed point of w, let $w' = 0$ and we get

$$\Omega_\Lambda = 3(1+w)/\lambda^2 \tag{3}$$

and further $\Omega'_\Lambda = -2 \times (1+w)\lambda'/\lambda^3$ or equivalently $\Omega'_\Lambda/\Omega_\Lambda = -2\lambda'/\lambda$, where $'$ denotes the derivative with respect to $\ln a$. Applying equations 2, we obtain

$$-3(w - w_m)(1 - \Omega_\Lambda) = 2\sqrt{3(1+w)\Omega_\Lambda}(\Gamma - 1)\lambda. \tag{4}$$

In the current universe, we set the initial conditions $\Omega_\Lambda(a_0) = 0.7$ and $w_m = 0$ and solve $\lambda(a_0)$, $w(a_0)$ from Eqs 3 and 4 in the tracking freezing models, $V = M^{4+p}Q^{-p}$ $(p = 1, 2, 4)$. In this case $\Gamma = 1 + 1/p > 1$ and the initial conditions are: $(p = 1, w(a_0) = -0.69, \lambda(a_0) = 1.15)$, $(p = 2, w(a_0) = -0.52, \lambda(a_0) = 1.43)$, $(p = 4, w(a_0) = -0.36, \lambda(a_0) = 1.66)$. For the early universe that is dominated by radiation, we assume $\Omega_\Lambda(a_0) \approx 0.01$ as a fiducial value and $w_m = 1/3$. The initial conditions become: $(p = 1, w(a_0) = -0.56, \lambda(a_0) = 11.49)$, $(p = 2, w(a_0) = -0.34, \lambda(a_0) = 14.07)$, $(p = 4, w(a_0) = -0.11, \lambda(a_0) = 16.34)$. Using these parameters, we show the $w' - w$ relation, the evolutions of $w(a)$ and $\Omega_\Lambda(a)$ in Figs. 1 and 2, respectively.

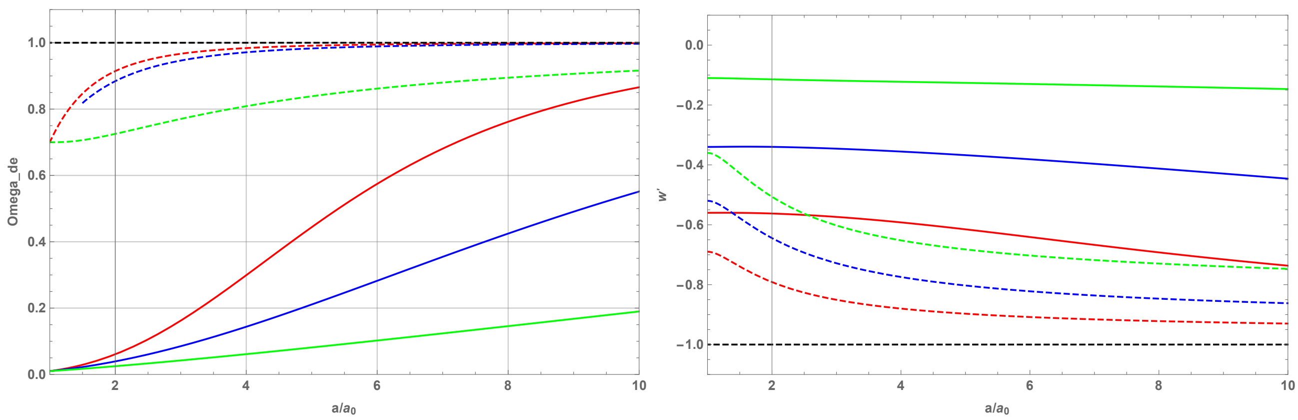

Figure 2: The evolutions of $\Omega_\Lambda(a)$ (left panel) and $w(a)$ (right panel) in a freezing model with the potential $V = M^{4+p}Q^{-p}$ ($p = 1, 2, 4$). Red, blue and greed lines correspond to $p = 1,\ 2,\ 4$, while dashed and solid lines are calculated in the present universe ($\Omega_\Lambda = 0.7, w_m = 0$) and radiation-dominated universe ($\Omega_\Lambda = 0.01, w_m = 1/3$), respectively.

4 Massive Quintessence

Assume $V(Q) = \frac{1}{2}M^2Q^2$, then we have

$$\lambda = \sqrt{\frac{1}{8\pi}}M_{pl}\frac{2}{Q} = \sqrt{\frac{1}{\pi}}\frac{M_{pl}M}{\sqrt{2V}}$$

Using $V = \frac{1-w}{2}\Omega\rho_c$, $\rho_c = \frac{\rho_m}{\Omega_m}$, $1 = \Omega + \Omega_m$ and $\rho_m = \frac{\rho_{m,0}}{a^{3(1+w_m)}}$. Combining this gives

$$\lambda = \sqrt{\frac{1}{\pi}}\frac{M_{pl}M}{\sqrt{\rho_{m,0}}}\sqrt{\frac{\frac{1}{\Omega}-1}{1-w}}e^{\frac{3}{2}(1+w_m)\log(a)}$$

From a given w and w' we can directly calculate M. Assuming that $1-w$ is small, we find

$$M = \frac{\sqrt{\pi}w'}{(1-w)\sqrt{1-\Omega}}\frac{\sqrt{\rho_{m,0}}}{M_{pl}} \approx 10^{-32}\frac{w'}{1-w}\frac{\text{eV}}{c^2}$$

Which means that either $1-w$ is increadibly small or the quintessence mass introduces a new hierarchy problem in physics. In fact, we quickly see that $\frac{V}{V'}$ has to be of order Planck mass, resulting in extremely flat potentials.This term will be proportional to $\frac{w'}{1-w}$ so a interesting measure of quality might be $\frac{1}{\sigma(\frac{w'}{1-w})}$, which can be made explicit as $\frac{(1-w)^2}{\sqrt{\sigma(w')^2(1-w)^2+w'^2\sigma(w)^2}}$.

Combining the expression for λ in this section and the equations for w, Ω_Λ, we obtain the evolution of $w(a)$ and the track in $w - w'$ space, as shown in Fig. 3. Here, we introduce one mass parameter $j = \frac{MM_p}{\pi\sqrt{\rho_{m,0}}}$ and it is in the order of unity (see the left panel of Fig. 3). A relatively higher M is favored to accelerate the growth of w and to reach the upper limit of w' (see the right panel).

5 Constraints on w and $w_a(w')$

If we know the values of w_0 and w' or w_a, the equation of state in a flat universe can be written as

$$p/\rho \simeq w_0 + w_a(a - a_0) = w_0 + w_a\frac{z}{1+z}.$$

Combining the equation of state with the Friedmann equation, we obtain

$$H^2 = H_0^2[\Omega_m(1+z)^3 + \Omega_\Lambda^{3(1+w)}]. \tag{5}$$

Parameters in this model are linked to observations through angular-diameter and/or luminosity distances. In practice, supernovae, gamma-ray bursts (GRBs) and gravitational lensing systems are used to constrain w_0 and w_a. Huterer and Peiris (2007) showed the constraints of $w_0 - w_a$

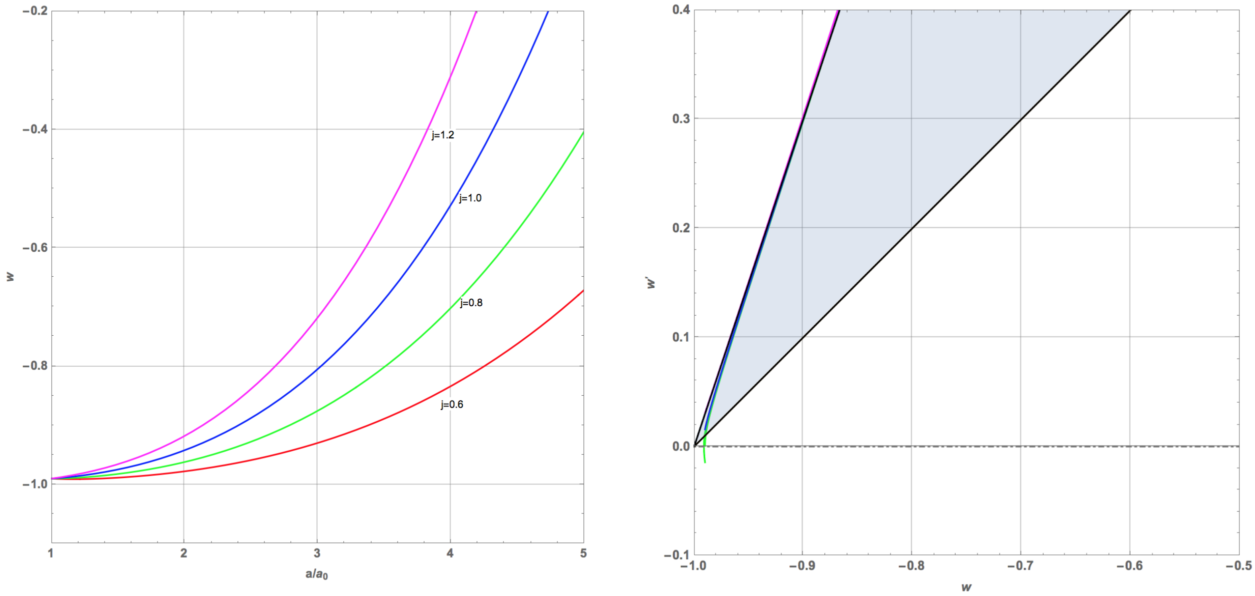

Figure 3: Evolution of $w(a)$ (left) and the track in $w-w'$ space (right) under $V=\frac{1}{2}M^2Q^2$. In this calculation we assume $\Omega_\Lambda(a_0)=0.7,\ w(a_0)=0.99$.